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Technical Notes

Bending Vibration of Rotating Tapered Cantilevers by Integral Equation Method

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Nomenclature

EI	=	flexural stiffness
F, f	=	centrifugal force and dimensionless centrifugal force
I, I_0	=	area moment of inertia and dimensionless area moment of inertia
L	=	beam length
M	=	bending moment
m, m_0	=	distributed mass and dimensionless distributed mass
Q	=	shear force
R, r	=	hub radius and dimensionless hub radius
W	=	amplitude of vibration
w	=	deflection
β	=	taper ratio
η	=	dimensionless rotational speed
λ	=	dimensionless natural frequency
ξ	=	normalized axial coordinate
Ω	=	rotational speed

I. Introduction

NONUNIFORM beams are frequently encountered structural components and have been widely employed in many fields, such as mechanical, civil, and aerospace engineering. Rotating structural elements are often designed to have different aspect ratios and tapered geometry in order to arrive at an optimum distribution of weight and strength. A particular area of interest for these structural elements is their vibrational behavior. A relatively long rotating cantilever beam can be modeled as an Euler–Bernoulli beam [1,2]. According to this theory, the governing equation for transverse bending vibration is a fourth-order differential equation with variable coefficients; then, getting an analytical solution is not easy, with the exception of some particular cases with the help of special functions [3,4]. Consequently, it is much desired to gain the natural frequencies for general situations. For this purpose, many approaches have been formulated to determine natural frequencies, such as the finite element method [5–7], the Frobenius series method [8], and the dynamic stiffness method [9,10]. Recently, Huang and Li [11] presented a new analytic approach to solve dynamic problems of axially graded elastic beams with varying flexural stiffness and nonuniform cross section. Ozgumus and Kaya [12] and Ozdemir and Kaya [13] formulated the differential transform method to give a vibration analysis of a rotating tapered Euler–Bernoulli beam and Timoshenko beam. Gunda et al. analyzed the mechanical behaviors of rotating beams with the aid of hybrid stiff-string-polynomial basic functions [14].

In this Note, an integral equation method is presented to analyze free vibration of rotating nonuniform beams. Natural frequencies can be readily sought from the existence condition of a nontrivial solution of the resulting equation. The results are compared with the ones available in open literature.

II. Derivation of Integral Equation

Consider a nonuniform straight slender beam rotating with an angular velocity Ω about the rotational axis, as shown in Fig. 1, where the x axis is orientated in the longitudinal direction of the beam and the z axis is parallel to and located at an offset R from the rotational axis (Fig. 1). The length of the beam is denoted as L, and the hub radius is denoted as R. For a realistic rotating blade, the problem is quite complex, where not only the out-of-plane (flapping) and in-plane (lead-lag) vibrations but also shear deformation and rotatory inertia should be involved. In the present Note, for simplicity, only the flapping vibration of a slender beam is concerned, and the coupling between flapping and lead–lag vibrations, as well as torsional and extensional coupling, is disregarded. Moreover, the Euler–Bernoulli theory of beams is adopted where the effects of shear deformation and rotatory inertia have been neglected. This is adequate for the analysis of a slender rotating beam [5–10].

Therefore, a differential equation governing the free bending vibration of a rotating beam reads

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2}\right]-\frac{\partial}{\partial x}\left[F(x)\frac{\partial w}{\partial x}\right]+m(x)\frac{\partial^2 w}{\partial t^2}=0 \tag{1}$$

where w is the bending deflection, $EI(x)$ is the flexural stiffness, and $F(x)$ is the centrifugal force arising from the rotating action. Distributed mass is denoted as $m(x)$, and the centrifugal force $F(x)$ is then represented by

$$F(x)=\int_x^L m(x)(R+x)\Omega^2\,\mathrm{d}x$$

For harmonic vibration, we take $w(x,t)=W(\xi)e^{i\omega t}$, where W is the amplitude, ω is the angular frequency, and $\xi=x/L$. Equation (1) can be transformed into a normalized form:

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[I_0(\xi)\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2}\right]-\frac{\mathrm{d}}{\mathrm{d}\xi}\left[f(\xi)\frac{\mathrm{d}W}{\mathrm{d}\xi}\right]-\lambda^2 m_0(\xi)W=0 \tag{2}$$

where we have introduced the following dimensionless variables:

$$\eta^2=\frac{m\Omega^2L^4}{EI},\qquad \lambda^2=\frac{m\omega^2L^4}{EI},\qquad r=\frac{R}{L},\qquad I_0(\xi)=\frac{I(x)}{I}$$
$$m_0(\xi)=\frac{m(x)}{m},\qquad f(\xi)=\eta^2\int_\xi^1 m_0(\xi)(r+\xi)\,\mathrm{d}\xi \tag{3}$$

In the preceding equation, I and m stand for the corresponding values of $I(x)$ and $m(x)$ at some reference positions, $I(0)$ and $m(0)$, respectively. Additionally, the bending moment M and shear force Q can be expressed in terms of $W(\xi)$ and its derivatives, suppressing the time factor $e^{i\omega t}$ as follows:

$$M(\xi)=-\frac{EI}{L^2}I_0(\xi)\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}$$

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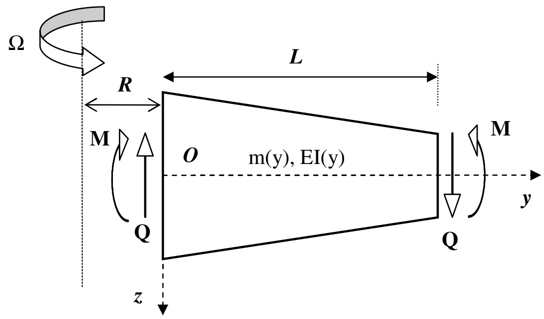

Fig. 1 Schematic of a rotating tapered beam along with the corresponding coordinate and the sign convention.

$$Q(\xi) = -\frac{EI}{L^3}\{\frac{\mathrm{d}}{\mathrm{d}\xi}[I_0(\xi)\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2}] - f(\xi)\frac{\mathrm{d}W}{\mathrm{d}\xi}\}$$

Our approach is to transform the differential equation (2) into an integral equation (see, e.g., [15]). To this end, we integrate both sides of Eq. (2) twice, with respect to ξ, from zero to ξ, yielding

$$I_0(\xi)\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2} - f(\xi)W(\xi) + \int_0^\xi f'(s)W(s)\,\mathrm{d}s - \lambda^2\int_0^\xi (\xi - s)m_0(s)W(s)\,\mathrm{d}s = C_1\xi + C_2 \tag{4}$$

where $C_j (j = 1, 2)$ are unknown constants, and the prime denotes differentiation with respect to the argument. In what follows, we consider a cantilever beam. In this case, it is clear that the boundary conditions at the free end ($\xi = 0$) and the built-in end ($\xi = 1$) can be stated as $W(0) = 0$, $W'(0) = 0$, $M(1) = 0$, and $Q(1) = 0$. Now, we set $\xi = 1$ in Eq. (4). Using the preceding conditions, we get

$$C_1 = -\lambda^2\int_0^1 m_0(s)W(s)\,\mathrm{d}s \tag{5}$$

$$C_2 = \int_0^1 f'(s)W(s)\,\mathrm{d}s + \lambda^2\int_0^1 s m_0(s)W(s)\,\mathrm{d}s \tag{6}$$

Furthermore, we integrate both sides of Eq. (4) twice with respect to ξ and obtain an integral equation as follows:

$$I_0(\xi)W(\xi) + \int_0^1 K(\xi, s)W(s)\,\mathrm{d}s - \lambda^2\int_0^1 H(\xi, s)W(s)\,\mathrm{d}s = 0 \qquad 0 \le \xi \le 1 \tag{7}$$

where

$$K(\xi, s) = \begin{cases} (\xi - s)I_0''(s) - 2I_0'(s) - (\xi - s)f(s) + \frac{1}{2}(\xi - s)^2 f'(s) - \frac{1}{2}\xi^2 f'(s), & 0 \le s \le \xi, \\ -\frac{1}{2}\xi^2 f'(s), & \xi < s \le 1 \end{cases} \tag{8}$$

$$H(\xi, s) = \begin{cases} \frac{1}{6}s^2(3\xi - s)m_0(s), & 0 \le s \le \xi, \\ \frac{1}{6}\xi^2(3s - \xi)m_0(s), & \xi < s \le 1 \end{cases} \tag{9}$$

It is noted that, for an Euler–Bernoulli beam with other supports such as simply supported ends, pinned–clamped ends, etc., similar integral equations can also be obtained by using boundary conditions of end supports, which are omitted here to save space. In the case of $F(x) = 0$, the details can be found in [11].

III. Numerical Results and Discussion

We have derived an integral equation through transforming the differential equation (2) subjected to the associated boundary conditions. An advantage of such a transformation is that the natural frequencies of the free bending vibration of rotating nonuniform beams can be easily calculated by seeking the eigenvalue of the resulting equation. Therefore, the drawback of the method of directly solving the governing differential equation with variable coefficients in Eq. (2) may be effectively overcome.

Now, we focus our attention on solving the eigenvalues of the resulting integral equation. Here, we use the expansion method to determine the eigenvalues of Eq. (7). We expand $W(\xi)$ as infinite power series in ξ, and then we truncate it as a finite sum. Thus, the unknown $W(\xi)$ can be approximately represented as

$$W(\xi) = \sum_{n=0}^{N} c_n\xi^n, \qquad 0 \le \xi \le 1 \tag{10}$$

where c_n are unknown coefficients, and N is a certain positive integer. It is expected that the solution based on our approach can reach any desired accuracy only if N is chosen large enough. Here, we take $N = 10$.

Inserting Eq. (10) into the resulting integral equation leads to

$$\sum_{n=0}^{N} c_n\xi^n I_0(\xi) + \sum_{n=0}^{N} c_n\int_0^1 K(\xi, s)s^n\,\mathrm{d}s - \lambda^2\sum_{n=0}^{N} c_n\int_0^1 H(\xi, s)s^n\,\mathrm{d}s = 0 \tag{11}$$

We multiply both sides of Eq. (11) by ξ^m and then integrate with respect to ξ between zero and one, yielding a system of linear algebraic equations in c_n:

$$\sum_{n=0}^{N}(i_{mn} + k_{mn} - \lambda^2 h_{mn})c_n = 0, \qquad m = 0, 1, 2, \dots, N \tag{12}$$

with

$$i_{mn} = \int_0^1 \xi^{m+n}I_0(\xi)\,\mathrm{d}\xi, \qquad k_{mn} = \int_0^1\int_0^1 K(\xi, s)\xi^m s^n\,\mathrm{d}s\,\mathrm{d}\xi$$
$$h_{mn} = \int_0^1\int_0^1 H(\xi, s)\xi^m s^n\,\mathrm{d}s\,\mathrm{d}\xi \tag{13}$$

To obtain a nontrivial solution of the resulting homogeneous system, the determinant of the coefficient matrix of the system has to vanish. Accordingly, we obtain a characteristic equation in λ^2:

$$\det(i_{mn} + k_{mn} - \lambda^2 h_{mn}) = 0 \tag{14}$$

First, let us consider a typical case, namely,

$$m_0(\xi) = (1 - \beta\xi)^n, \qquad I_0(\xi) = (1 - \beta\xi)^{n+2} \tag{15}$$

In general, n is a parameter and often takes positive integers, and β is a taper ratio satisfying $0 \le \beta \le 1$. This case is of much importance in practical application. For example, when $n = 1$, the beam is of constant width and linearly varying depth, whereas when $n = 2$, the width and depth of the beam are both linearly varying. When $\beta = 0$,

Table 1 Dimensionless natural frequencies λ_j of a rotating beam with various rotational speeds[a]

	λ_1			λ_2			λ_3		
η	Present	[10]	[5,7]	Present	[10]	[5,7]	Present	[10]	[5,7]
0	3.82378	3.82379	3.8238	18.3173	18.3173	18.3173	47.2648	47.2648	47.2648
1	3.98662	3.98661	3.9866	18.4740	18.4740	18.4740	47.4173	47.4173	47.4173
2	4.43680	4.43680	4.4368	18.9366	18.9366	18.9366	47.8716	47.8717	47.8716
3	5.09267	5.09267	5.0927	19.6839	19.6839	19.6839	48.6190	48.6190	48.6190
4	5.87876	5.87877	5.8788	20.6852	20.6851	20.6852	49.6456	49.6456	49.6456
5	6.74340	6.74340	6.7434	21.9053	21.9053	21.9053	50.9338	50.9338	50.9338
6	7.65514	7.65514	7.6551	23.3093	23.3093	23.3093	52.4633	52.4632	52.4633
7	8.59558	8.59557	8.5956	24.8647	24.8647	24.8647	54.2124	54.2124	54.2124
8	9.55396	9.55396	9.5540	26.5437	26.5437	26.5437	56.1595	56.1595	56.1595
9	10.5239	10.5239	10.5239	28.3227	28.3227	28.3227	58.2833	58.2833	58.2833
10	11.5015	11.5015	11.5015	30.1827	30.1827	30.1827	60.5639	60.5639	60.5639

[a]The parameters are as follows: $m_0(\xi) = 1 - 0.5\xi$, $I_0(\xi) = (1 - 0.5\xi)^3$, and $r = 0$.

Table 2 Dimensionless natural frequencies λ_j for a rotating beam[a]

	λ_1			λ_2			λ_3		
N	$\eta = 0$	$\eta = 5$	$\eta = 10$	$\eta = 0$	$\eta = 5$	$\eta = 10$	$\eta = 0$	$\eta = 5$	$\eta = 10$
5	3.82380	6.74338	11.5014	18.3018	21.8830	30.1570	46.8252	50.3449	59.6725
8	3.82378	6.74340	11.5015	18.3173	21.9053	30.1827	47.2655	50.9338	60.5635
10	3.82378	6.74340	11.5015	18.3173	21.9053	30.1827	47.2648	50.9338	60.5639
12	3.82378	6.74340	11.5015	18.3173	21.9053	30.1827	47.2648	50.9338	60.5639

[a]The parameters are as follows: $m_0(\xi) = 1 - 0.5\xi$, $I_0(\xi) = (1 - 0.5\xi)^3$, and $r = 0$.

the tapered beam reduces to a uniform beam, whereas when $\beta = 1$, the beam tapers to a point, which becomes cone, pyramid, or wedge beams.

Obtained λ values are tabulated in Table 1 with $\beta = 0.5$, $r = 0$, and various rotational speeds. It is worth noting that our numerical results for $n = 1$ are identical to those in [5,7], where a finite element with 15-order polynomial functions as the interpolation functions and a single spectral finite element with the first 80 terms of the Frobenius power series were used to obtain numerical results, respectively. To validate rapid convergence of the present method, in the same mass density and bending stiffness as considered previously, we first calculate three dimensionless natural frequencies λ_j for several different N values. Evaluated results are presented in Table 2. Clearly, only a few terms are needed in the suggested method, and it gives the same accuracy up to four decimal places as the results obtained in [7], where 80 terms of the Frobenius series were needed. This indicates that the method is more efficient than the Frobenius series expansion method. For this case, once λ_j are determined, nonvanishing unknown c_n can be found. Consequently, the first three mode shapes of a nonuniform cantilever can be obtained and shown in Fig. 2.

Table 3 Dimensionless natural frequencies λ_j of a rotating beam with various rotational speeds[a]

	λ_1		λ_2		λ_3	
η	Present	[10]	Present	[10]	Present	[10]
0	4.62515	4.62515	19.5476	19.5476	48.5789	48.5789
1	4.76405	4.76405	19.6803	19.6803	48.7073	48.7073
2	5.15641	5.15641	20.0734	20.0733	49.0906	49.0906
3	5.74578	5.74578	20.7121	20.7121	49.7226	49.7227
4	6.47262	6.47262	21.5749	21.5749	50.5939	50.5938
5	7.29014	7.29014	22.6360	22.6360	51.6918	51.6918
6	8.16629	8.16630	23.8684	23.8684	53.0019	53.0018
7	9.08035	9.08036	25.2461	25.2461	54.5081	54.5082
8	10.01925	10.0192	26.74544	26.7454	56.19411	56.1941
9	10.97472	10.9747	28.34587	28.3459	58.04333	58.0434
10	11.94149	11.9415	30.02989	30.0299	60.03986	60.0399
		8.93378[b]		25.3048[b]		55.0124[b]

[a]The parameters are as follows: $m_0(\xi) = (1 - 0.5\xi)^2$, $I_0(\xi) = (1 - 0.5\xi)^4$, and $r = 0$.
[b]Results taken from [6], the case when $n = 2$ and $\eta = 10$.

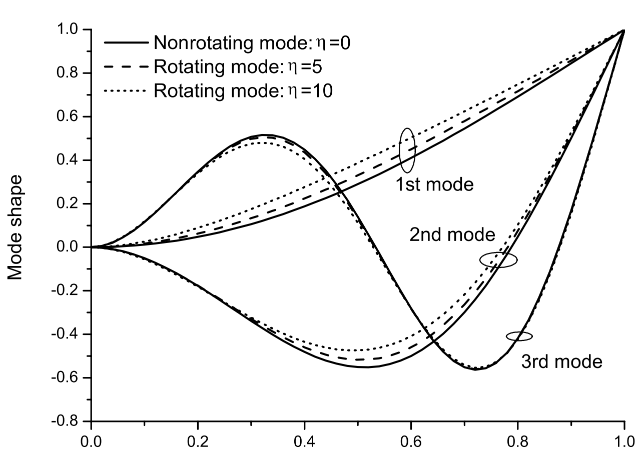

Fig. 2 First three mode shapes of a nonuniform cantilever beam with $m_0(\xi) = 1 - 0.5\xi$ and $I_0(\xi) = (1 - 0.5\xi)^3$.

In the case of $n = 2$, the corresponding results are presented in Table 3. Our numerical results agree very well with those in [10]. This also indicates the validation of [10]. As pointed out in [10], the results reported by [6] have considerable differences (around 25% in the fundamental frequencies) as compared with ours and those in [10], which are displayed in Table 3. Therefore, for this example, the present approach is still very effective. In addition, the rotational speed increases the natural frequencies of free bending vibration.

In what follows, the effects of β on λ are examined for three different rotational speeds. Obtained results are listed in Tables 4 and 5 for $n = 1$ and 2, respectively. It should be noted that our results for $\beta = 1$ can be directly computed. However, the previous results obtained in [10] were evaluated by extrapolation based on the parabolic limit of the two sets of results obtained using $\beta = 0.99$ and 0.995. When rotational speed is equal to zero, the natural frequencies of the free vibration of wedge and cone beams were calculated in [3], which are given in Tables 4 and 5, whereas when $\beta = 0$, the natural frequencies of the free vibration of uniform beams were also given in

Table 4 Dimensionless natural frequencies λ_j of a rotating beam with various rotational speeds[a]

η	β	λ_1 Present	λ_1 [10]	λ_2 Present	λ_2 [10]	λ_3 Present	λ_3 [10]
0	0	3.51602	——	22.0345	——	61.6972	——
	0.2	3.60827	3.60827	20.6210	20.6210	56.1923	56.1923
	0.4	3.73708	3.73708	19.1138	19.1138	50.3537	50.3537
	0.6	3.93428	3.93428	17.4879	17.4878	44.0248	44.0248
	0.8	4.29250	4.29249	15.7428	15.7427	36.8861	36.8846
	1	5.31510	5.27968[b]	15.2072	15.1072[b]	30.0202	29.8330[b]
			5.31510[c]		15.2072[c]		30.0198[c]
5	0	6.44954	——	25.4461	——	65.2050	——
	0.2	6.53913	6.53913	24.0961	24.0961	59.7504	59.7504
	0.4	6.66206	6.66206	22.6612	22.6612	53.9790	53.9789
	0.6	6.84537	6.84537	21.1207	21.1207	47.7478	47.7478
	0.8	7.16281	7.16281	19.4849	19.4848	40.7740	40.7725
	1	7.95671	7.93186[b]	18.9590	18.8783[b]	34.2094	34.0448[b]
10	0	11.2023	——	33.6404	——	74.6493	——
	0.2	11.2950	11.2950	32.3325	32.3325	69.2537	69.2537
	0.4	11.4200	11.4200	30.9292	30.9292	63.5597	63.5598
	0.6	11.6023	11.6023	29.4013	29.4013	57.4378	57.4379
	0.8	11.9055	11.9055	27.7441	27.7441	50.6378	50.6370
	1	12.5846	——	27.0213	——	44.3132	——

[a]The parameters are as follows: $m_0(\xi) = 1 - \beta\xi$, $I_0(\xi) = (1 - \beta\xi)^3$, and $r = 0$.
[b]Results evaluated from a limitation procedure.
[c]Results taken from [3].

Table 5 Dimensionless natural frequencies λ_j of a rotating beam with various rotational speeds[a]

η	β	λ_1 Present	λ_1 [10]	λ_2 Present	λ_2 [10]	λ_3 Present	λ_3 [10]
0	0	3.51602	3.5160[b]	22.0345	22.0345[b]	61.6972	61.6972[b]
	0.2	3.85512	3.85511	21.0567	21.0568	56.6303	56.6303
	0.4	4.31878	4.31878	20.0500	20.0500	51.3346	51.3346
	0.6	5.00903	5.00904	19.0649	19.0649	45.7384	45.7384
	0.8	6.19639	6.19639	18.3856	18.3855	39.8361	39.8336
	1	8.71926	8.66108[c]	21.1457	21.0039[c]	38.4547	38.1905[c]
			8.71926[d]		21.1457[d]		38.4538[d]
5	0	6.44954	6.4495[b]	25.4461	25.4461[b]	65.2050	65.2050[b]
	0.2	6.69693	6.69693	24.3478	24.3478	59.9763	59.9763
	0.4	7.04980	7.04980	23.2089	23.2088	54.5230	54.5230
	0.6	7.59724	7.59724	22.0803	22.0803	48.7798	48.7797
	0.8	8.56998	8.56998	21.2522	21.2521	42.7578	42.7558
	1	10.6470	10.3284[c]	23.6723	22.8804[c]	41.2118	40.2096[c]
10	0	11.2023	11.2023[b]	33.6404	33.6404[b]	74.6493	74.6493[b]
	0.2	11.4188	11.4188	32.2499	32.2499	68.9817	68.9817
	0.4	11.7295	11.7295	30.7818	30.7818	63.0886	63.0887
	0.6	12.2119	12.2119	29.2867	29.2867	56.9143	56.9143
	0.8	13.0550	13.0550	28.0752	28.0752	50.4963	50.4956
	1	14.7417		29.8678		48.4782	

[a]The parameters are as follows: $m_0(\xi) = (1 - \beta\xi)^2$, $I_0(\xi) = (1 - \beta\xi)^4$, $r = 0$.
[b]Results taken from [4].
[c]Results evaluated from a limitation procedure.
[d]Results taken from [3].

[4], which are denoted by [b] in Table 5 by comparison, we find that our results are in excellent agreement with those obtained in [3,4,10]. Moreover, from Tables 4 and 5, we further observe that with an increase in the taper ratio β, the fundamental frequency is always raised, and the third-order natural frequency always drops down.

As a second example, we consider a tapered beam with linearly varying mass distribution and bending stiffness; that is,

$$m_0(\xi) = 1 - 0.8\xi, \qquad I_0(\xi) = 1 - 0.95\xi \tag{16}$$

Such structural beams were employed in the design of windmill turbine blades [8]. For this case, the natural frequencies are computed and listed in Table 6. In the case when the nondimensional hub radius $r \neq 0$, we also examined the effect of r on λ for different rotational speeds (see Table 7). In Table 7, the numerical results in [9] were obtained by approximately discretizing the tapered beam into 50 uniform stepped beams.

Finally, in order to show the present method can be employed to tackle more complicated beam cases, for simplicity, a dual tapered cantilever with mass distribution and bending stiffness,

Table 6 Dimensionless natural frequencies λ_j of a rotating beam with various rotational speeds[a]

η	λ_1 Present	λ_1 [7]	λ_1 [8]	λ_2 Present	λ_2 [7]	λ_2 [8]	λ_3 Present	λ_3 [7]	λ_3 [8]
0	5.2738	5.2738	5.2738	24.0041	24.0041	24.0041	59.9701	59.9708	59.9708
1	5.3903	5.3903	5.3903	24.1070	24.1069	24.1069	60.0696	60.0703	60.0696
2	5.7249	5.7249	5.7249	24.4130	24.4129	24.4130	60.3669	60.3676	60.3669
3	6.2402	6.2402	6.2402	24.9149	24.9148	24.9149	60.8590	60.8598	60.8590
4	6.8928	6.8928	6.8928	25.6013	25.6013	25.6013	61.5412	61.5420	61.5412
5	7.6443	7.6443	7.6443	26.4581	26.4580	26.4581	62.4069	62.4078	62.4069
6	8.4653	8.4653	8.4653	27.4693	27.4692	27.4693	63.4483	63.4494	63.4483
7	9.3347	9.3347	9.3347	28.6185	28.6184	28.8894	64.6566	64.6579	64.6566
8	10.2379	10.2379	10.2379	29.8894	29.8893	29.8894	66.0223	66.0238	66.0222
9	11.1650	11.1651	11.1650	31.2669	31.2667	31.2669	67.5352	67.5370	67.5351
10	12.1092	12.1092	12.0192	32.7369	32.7367	32.7369	69.1852	69.1875	69.1851

[a]The parameters are as follows: $m_0(\xi) = 1 - 0.8\xi$, $I_0(\xi) = 1 - 0.95\xi$, and $r = 0$.

Table 7 Dimensionless natural frequencies λ_j of a rotating beam with various rotational speeds[a]

r	η	λ_1 Present	λ_1 [9]	λ_1 [8]	λ_2 Present	λ_2 [9]	λ_2 [8]	λ_3 Present	λ_3 [9]	λ_3 [8]
0	0	5.27376	5.2725	5.2738	24.0041	23.995	24.004	59.9701	59.943	59.970
	1	5.39031	——	——	24.1070	——	——	60.0696	——	——
	5	7.64434	7.6434	7.6443	26.4581	26.450	26.458	62.4069	62.380	62.407
1	1	5.55058	5.5493	5.5507	24.2498	24.241	24.250	60.2134	60.186	60.214
	5	10.0828	10.082	10.083	29.5340	29.525	29.535	65.7635	65.737	67.765
5	1	6.14937	6.1482	6.1494	24.8129	24.804	24.813	60.7850	60.758	60.785
	5	16.5118	16.510	16.512	39.4122	39.404	39.413	77.6019	77.575	77.602

[a]The parameters are as follows: $m_0(\xi) = 1 - 0.8\xi$ and $I_0(\xi) = 1 - 0.95\xi$.

Table 8 Dimensionless natural frequencies λ_j of a cantilevered beam with discontinuities in mass density and bending stiffness and with $\eta = 1$ and $r = 0.05$

Mode	Present ($N = 7$)	Present ($N = 10$)	Present ($N = 13$)	[7]	[16]
1	1.07973	1.06478	1.06134	1.0660075	1.0660084
2	2.61370	2.58866	2.58351	2.5906956	2.5906956
3	4.74837	4.72304	4.72064	4.7187589	4.7187589
4	7.68139	7.71231	7.71625	7.6691747	7.6691747

$$m_0(\xi) = \begin{cases} 1, & 0 \le \xi \le 0.2 \\ 5(1 - 0.5\xi), & 0.2 < \xi \le 1 \end{cases}$$

$$I_0(\xi) = \begin{cases} 0.00146, & 0 \le \xi \le 0.2 \\ 0.0146(1 - 1.5\xi + 0.75\xi^2), & 0.2 < \xi \le 1 \end{cases}$$

is analyzed. In calculation, the dimensionless hub offset length $r = 0.05$ and dimensionless rotating speed $\eta = 1$ are assumed. The obtained results are displayed when taking $N = 7$, 10 in Table 8. From Table 8, by comparing our results when taking $N = 10$ with those given in [7,16], one can find that the error does not exceed 0.6%. For the first three order natural frequencies, the error is less 0.1%. Therefore, our results are in satisfactory consistency with those given in [7,16]. It should be pointed out that our results are evaluated only when taking $N = 10$ [i.e., the first 11 terms in Eq. (10)], whereas in [7], 200 terms in the Frobenius series were included. It is worth noting that, for this case, when N continues to rise, the convergency of numerical results becomes very slow, and even deteriorates. Therefore, the efficiency of this method for discontinuous rotating beams is not very good, and this may be attributed to the discontinuity of the material properties, because the flexural rigidity in the present analysis is a differentiable function.

IV. Conclusions

Instead of directly solving the fourth-order governing differential equation with variable coefficients, the integral equation method has been proposed to deal with the free bending vibration of rotating nonuniform cantilever beams. This method is capable of treating Euler–Bernoulli beams with an arbitrarily continuously varying cross section, such as polynomial functions, or with linearly varying depth or/and width and other tapered beams. Moreover, the obtained results have high accuracy as compared with other numerical results.

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